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Bäcklund transformations for new integrable hierarchies related to the polynomial Lie algebra $\mathfrak{gl}_{\infty}^{(n)}$

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Abstract

In a recent paper [P. Casati, G. Ortenzi, New integrable hierarchies from vertex operator representations of polynomial Lie algebras, J. Geom. Phys. 56 (3) (2006) 418–449] Casati and Ortenzi gave a representation-theoretic interpretation of recently discovered coupled soliton equations, which were described by e.g. R. Hirota, X. Hu, X. Tang [A vector potential KdV equation and vector Ito equation: Soliton solutions, bilinear Bäcklund transformations and Lax pairs, J. Math. Anal. Appl. 288 (1) (2003) 326–348. [3]], S. Kakei [Dressing method and the coupled KP hierarchy, Phys. Lett. A 264 (6) (2000) 449–458. [6]] and S.Yu. Sakovich [A note in the Painlevé property of coupled KdV equation, arXiv:nlin.SI/0402004. [7]]. Casati and Ortenzi use vertex operators for these Lie algebras and a boson–fermion type of correspondence to get a hierarchy of coupled Hirota bilinear equations. In this paper we reformulate the Hirota bilinear description for the Lie algebra $\mathfrak{gl}_{\infty}^{(n)}$ and obtain a bilinear identity for matrix wave functions. From that it is straightforward to deduce the Sato–Wilson, Lax and Zakharov–Shabat equations. Using these wave functions and standard calculus with vertex operators we obtain elementary Bäcklund–Darboux transformations.

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1. The polynomial Lie algebra $\mathfrak{gl}_{\infty}^{(n)}$

We follow the description of Casati and Ortenzi [1] and introduce 'polynomial Lie algebras'. For any positive integer n, let $\mathbb{C}^{(n)}(\lambda)$ be the commutative algebra $\mathbb{C}[\lambda]/(\lambda)^{n+1}$. If $\mathfrak{g}$ is a Lie algebra, we introduce the Lie algebra

$$\mathfrak{g}^{(n)} = \mathfrak{g} \otimes \mathbb{C}^{(n)}(\lambda). \tag{1.1}$$

We identify $g^{(n)}$ with the Lie algebra of polynomial maps $\mathbb{C}^{(n)}(\lambda) \to \mathfrak{g}$. An element $X(\lambda)$ in $\mathfrak{g}^{(n)}$ will be viewed as the mapping $X : \mathbb{C}^{(n)}(\lambda) \to \mathfrak{g}$, $X(\lambda) = \sum_{k=0}^{n} X_k \lambda^k$. The Lie bracket of two elements in $\mathfrak{g}^{(n)}$, $X(\lambda) = \sum_{k=0}^{n} X_k \lambda^k$

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and $Y(\lambda)=\sum_{k=0}^{n} Y_k\lambda^k$ will be given by

$$[X(\lambda), Y(\lambda)] = \sum_{k=0}^{n}\left(\sum_{j=0}^{k}[X_j, Y_{k-j}]_{\mathfrak{g}}\right)\lambda^k, \tag{1.2}$$

where $[\cdot,\cdot]_{\mathfrak{g}}$ is the Lie bracket defined on $\mathfrak{g}$.

We now take for $\mathfrak{g}$ the infinite dimensional Lie algebra $\mathfrak{gl}_\infty$ (see e.g. [4]), hence $\mathfrak{gl}_\infty^{(n)}$ is the infinite dimensional Lie algebra given by the tensor product

$$\mathfrak{gl}_\infty^{(n)} = \mathfrak{gl}_\infty \otimes \mathbb{C}^{(n)}(\lambda). \tag{1.3}$$

Then $\mathfrak{gl}_\infty^{(n)}$ is the Lie algebra given by the linear span of the basis $\{E_{ij}^k\}_{j,i\in\mathbb{Z},k=0,\ldots,n}$, with $E_{ij}^k := E_{ij}\lambda^k$. The Lie bracket is given by

$$\left[E_{ij}^k, E_{lm}^s\right] = \begin{cases}\delta_{jl}E_{im}^{k+s} - \delta_{im}E_{lj}^{k+s} & \text{if } k+s\le n,\\ 0 & \text{otherwise.}\end{cases} \tag{1.4}$$

Following [1] we define a representation of this algebra on $n+1$ copies of the an infinite wedge space:

$$F^{(n)} = \bigoplus_{i=0}^{n} F^i, \tag{1.5}$$

where the spaces F^i with $i=0,\ldots,n$ are copies of the semi-infinite wedge space F generated by the semi-infinite monomials

$$\underline{i}_1\wedge\underline{i}_2\wedge\cdots\wedge\underline{i}_j\wedge\ldots$$

where $\underline{i}_j$ with $i_j\in\mathbb{Z}$ are basis vectors of $\mathbb{C}^\infty$, such that

$$i_1 > i_2 > i_3 \quad\text{and}\quad i_j = i_{j-1}-1 \quad \text{for all } j\gg 0$$

(see [4] for more details). We think of elements in $F^{(n)}$ as $(n+1)$-dimensional column vectors with entries from F.

It will be convenient to define on $F^{(n)}$ the following operators $\psi_i^{+(k)}$ and $\psi_i^{-(k)}$ which act as

$$\begin{aligned}&\psi_i^{\pm(k)}(\underline{i}_{01}\wedge\underline{i}_{02}\wedge\ldots,\underline{i}_{11}\wedge\underline{i}_{12}\wedge\ldots,\ \ldots,\ \underline{i}_{n1}\wedge\underline{i}_{n2}\wedge\ldots)^{\mathrm{T}}\\ &\quad= (\underbrace{0,\ldots,0}_{k},\psi_i^{\pm}(\underline{i}_{01}\wedge\underline{i}_{02}\wedge\ldots),\psi_i^{\pm}(\underline{i}_{11}\wedge\underline{i}_{12}\wedge\ldots),\ldots,\psi_i^{\pm}(\underline{i}_{n-k,1}\wedge\underline{i}_{n-k,2}\wedge\ldots))^{\mathrm{T}},\end{aligned} \tag{1.6}$$

where the action of the operators $\psi_j^{\pm}$ is given by the usual formulas [4]:

$$\begin{aligned}\psi_j^{+}(\underline{i}_1\wedge\underline{i}_2\wedge\ldots) &= \begin{cases}0 & \text{if } j=i_s \text{ for some } s,\\ (-1)^s\underline{i}_1\wedge\cdots\wedge\underline{i}_s^{m}\wedge\underline{i}_{i_s+1}\wedge\ldots & \text{if } i_s>j>i_{s+1},\end{cases}\\ \psi_j^{-}(\underline{i}_1\wedge\underline{i}_2\wedge\ldots) &= \begin{cases}0 & \text{if } j\ne i_s \text{ for all } s,\\ (-1)^{s+1}\underline{i}_1\wedge\cdots\wedge\underline{i}_{s-1}\wedge\underline{i}_{i_s+1}\wedge\ldots & \text{if } j=i_s.\end{cases}\end{aligned} \tag{1.7}$$

One easily checks that for $j+k=\ell+m$:

$$\begin{aligned}&\psi_i^{+(k)}\psi_l^{+(j)} + \psi_l^{+(\ell)}\psi_i^{+(m)} = 0, \qquad \psi_i^{-(k)}\psi_l^{-(j)} + \psi_l^{-(\ell)}\psi_i^{-(m)} = 0,\\ &\psi_i^{+(k)}\psi_l^{-(j)} + \psi_l^{-(\ell)}\psi_i^{+(m)} = \begin{cases}\delta_{il}\Lambda^{j+k} & \text{if } j+k\le n,\\ 0 & \text{otherwise,}\end{cases}\end{aligned} \tag{1.8}$$

where Λ is the $n+1 \times n+1$-matrix

$$\Lambda = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

It is convenient to have the generating fields of these operators

$$\psi^{+(k)}(z) = \sum_{j\in\mathbb{Z}} \psi_j^{+(k)} z^{j-1}, \qquad \psi^{-(k)}(z) = \sum_{j\in\mathbb{Z}} \psi_j^{-(k)} z^{-j},$$

then the relations (1.8) can be rewritten as

$$\begin{aligned} &\psi^{+(k)}(y)\psi^{+(j)}(z) + \psi^{+(\ell)}(z)\psi^{+(m)}(y) = 0, \qquad \psi^{-(k)}(y)\psi^{-(j)}(z) + \psi^{-(\ell)}(z)\psi^{-(m)}(y) = 0, \\ &\psi^{+(k)}(y)\psi^{-(j)}(z) + \psi^{-(\ell)}(z)\psi^{+(m)}(y) = \begin{cases} \delta(y-z)\Lambda^{j+k} & \text{if } j+k \le n, \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \tag{1.9}$$

for all $j+k = \ell+m$, where

$$\delta(y-z) = y^{-1} \sum_{j\in\mathbb{Z}} \left(\frac{y}{z}\right)^j.$$

The space $F^{(n)}$ has a charge decomposition

$$F^{(n)} = \bigoplus_{m\in\mathbb{Z}} F_m^{(n)}$$

on letting

$$|m\rangle^i = (\underbrace{0,\ldots,0}_{i}, \underline{m} \wedge \underline{m-1} \wedge \underline{m-2} \wedge \ldots, 0, \ldots, 0)^{\mathrm{T}}$$

and have charge m in F^i and charge $(\psi_j^{\pm(k)}) = \pm j$.

We call $|0\rangle = |0\rangle^0$ the vacuum vector. It will also be useful to introduce the m-th vacuum vector

$$|m\rangle = |m\rangle^0 = (m \wedge m-1 \wedge m-2 \wedge \ldots, \underbrace{0,\ldots,0}_{n})^{\mathrm{T}}.$$

It is clear that

$$\psi_j^{+(k)}|m\rangle = 0 \quad \text{for } j \le m, \quad \psi_j^{-(k)}|m\rangle = 0 \quad \text{for } j > m, \quad k = 0,\ldots,n.$$

We can now define the representation ρ of $\mathfrak{gl}_\infty^{(n)}$ on $F^{(n)}$ by setting

$$\rho(E_{ij}^k) = \frac{1}{k+1} \sum_{l=0}^{k} \psi_i^{+(k-l)} \psi_j^{-(l)} \qquad i,j \in \mathbb{Z}, k = 0,\ldots,n. \tag{1.10}$$

It is obvious that the actions of the elements E_{ij}^k do not change the charge. One can easily show that $F_m^{(n)}$ is an irreducible $\mathfrak{gl}_\infty^{(n)}$-module. Using the boson–fermion correspondence as described by [1], see also [4], we can identify the space $F^{(n)}$ with the space

$$B^{(n)} = \mathbb{C}[q, q^{-1}, x_1, x_2, \ldots] \otimes \mathbb{C}^{n+1}.$$

Let σ be the isomorphism that transposes $F^{(n)}$ to $B^{(n)}$. Then

$$\sigma\psi^{\pm(k)}(z)\sigma^{-1} = q^{\pm 1} z^{\pm q\frac{\partial}{\partial q}} \exp(\pm\xi(x,z)) \exp(\mp\eta(x,z)) \otimes \Lambda^k,$$

where

$$\xi(x,z)=\sum_{j=1}^{\infty}x_jz^j,\qquad \eta(x,z)=\sum_{j=1}^{\infty}\frac{z^{-j}}{j}\frac{\partial}{\partial x_j}.$$

One also has

$$\sigma(F_m^{(n)})=B_m^{(n)}:=q^m\mathbb{C}[x_1,x_2,\dots]\otimes\mathbb{C}^{n+1}.$$

2. Wave functions and Lax equations

Casati and Ortenzi introduce in [1] the group $GL_\infty^{(n)}=GL_\infty\ltimes N_\infty^{(n)}$ where

$$N_\infty^{(n)}=\{I+X|\ X\in\mathfrak{gl}_\infty\otimes\lambda\mathbb{C}^{(n-1)}(\lambda)\}.$$

They show that for $m=0$, the element $\tau^{(m)}\in F_m^{(n)}$ is in the group orbit $\tau^{(m)}\in GL_\infty^{(n)}|m\rangle$ if and only if $\tau^{(m)}$ satisfies

$$\mathrm{Res}_z\sum_{k=0}^{n}\sum_{\ell=0}^{k}\frac{1}{k+1}\psi^{+(k-\ell)}(z)\tau^{(m)}\otimes_{\mathbb{C}^{(n)}(\lambda)}\psi^{-(\ell)}(z)\tau^{(m)}=0.\tag{2.1}$$

Of course the statement also holds for all $m\in\mathbb{Z}$. This equation once bosonized gives an infinite family of Hirota bilinear equations, see [1] for the details. We will bosonize here, but will not produce these Hirota bilinear equations. We will rather be interested in wave functions and the corresponding Lax equations. It will be convenient to write these equations in

$$q^m\mathbb{C}[x_1,x_2,\dots]\otimes q^m\mathbb{C}[x_1,x_2,\dots]\otimes\mathbb{C}^{(n)}(\lambda).$$

Then $\tau^{(m)}$ can be written as

$$\tau^{(m)}(x)=\sum_{j=0}^{n}\tau_j^{(m)}(x)\lambda^jq^m$$

and Eq. (2.1) becomes

$$\mathrm{Res}_z\sum_{p,q=0}^{n}\sum_{k=0}^{n}z^m e^{\xi(x,z)}e^{-\eta(x,z)}\tau_p^{(m)}(x)\otimes z^{-m}e^{-\xi(x,z)}e^{\eta(x,z)}\tau_q^{(m)}(x)\otimes\lambda^{p+q+k}=0,\tag{2.2}$$

which gives that for all $k=0,1,2,\dots,n$ one has

$$\mathrm{Res}_z\sum_{j=0}^{k}z^m e^{\xi(x,z)}e^{-\eta(x,z)}\tau_j^{(m)}(x)\otimes z^{-m}e^{-\xi(x,z)}e^{\eta(x,z)}\tau_{k-j}^{(m)}(x)=0.\tag{2.3}$$

We now rewrite this in a matrix form, introduce the tau matrix

$$T^{(m)}(x)=\begin{pmatrix}\tau_0^{(m)}(x)&0&0&\dots&0\\ \tau_1^{(m)}(x)&\tau_0^{(m)}(x)&0&\dots&0\\ \tau_2^{(m)}(x)&\tau_1^{(m)}(x)&\tau_0^{(m)}(x)&\dots&0\\ \vdots&\vdots&\vdots&\ddots&0\\ \tau_n^{(m)}(x)&\tau_{n-1}^{(m)}(x)&\tau_{n-2}^{(m)}(x)&\dots&\tau_0^{(m)}(x)\end{pmatrix},\tag{2.4}$$

and then

$$\mathrm{Res}_z\sum_{j=0}^{k}z^m e^{\xi(x,z)}e^{-\eta(x,z)}T^{(m)}(x)z^{-m}e^{-\xi(x',z)}e^{\eta(x',z)}T^{(m)}(x')=\mathbf{0}.\tag{2.5}$$

We forget the tensor symbol and write x' for the second component of the tensor product. This equation can also be written in terms of wave functions, let

$$\Phi^{\pm}(m,x,z)=z^{\pm m}e^{\pm\xi(x,z)}\begin{pmatrix} e^{\mp\eta(x,z)}\tau_0^{(m)} & 0 & 0 & \dots & 0\\ e^{\mp\eta(x,z)}\tau_1^{(m)} & e^{\mp\eta(x,z)}\tau_0^{(m)} & 0 & \dots & 0\\ e^{\mp\eta(x,z)}\tau_2^{(m)} & e^{\mp\eta(x,z)}\tau_1^{(m)} & e^{\mp\eta(x,z)}\tau_0^{(m)} & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & 0\\ e^{\mp\eta(x,z)}\tau_n^{(m)} & e^{\mp\eta(x,z)}\tau_{n-1}^{(m)} & e^{\mp\eta(x,z)}\tau_{n-2}^{(m)} & \dots & e^{\mp\eta(x,z)}\tau_0^{(m)} \end{pmatrix}. \tag{2.6}$$

Then (2.5) becomes

$$\mathrm{Res}_z\,\Phi^+(m,x,z)\Phi^-(m,x',z)=\mathbf{0}. \tag{2.7}$$

Now multiply both $\Phi^{\pm}(m,x,z)$ with the inverse of $T^{(m)}(x)$, i.e., we define

$$\Psi^{\pm}(m,x,z)=\left(T^{(m)}(x)\right)^{-1}\Phi^{\pm}(m,x,z)=\Phi^{\pm}(m,x,z)\left(T^{(m)}(x)\right)^{-1}. \tag{2.8}$$

Then (2.7) becomes

$$\mathrm{Res}_z\,\Psi^+(m,x,z)\Psi^-(m,x',z)=\mathbf{0}. \tag{2.9}$$

We call $\Psi^+(m,x,z)$ the m-th wave function or wave matrix and $\Psi^-(m,x,z)$ the m-th adjoint wave function/matrix. It will be convenient to write them as follows

$$\begin{aligned}\Psi^+(m,x,z)&=P^+(m,x,\partial)\partial^m e^{\xi(x,z)},\\ \Psi^-(m,x,z)&=P^-(m,x,\partial)(-\partial)^{-m}e^{-\xi(x,z)},\end{aligned} \tag{2.10}$$

as pseudo-differential operators in $\partial=\partial_{x_1}$ with coefficients of the form

$$\begin{pmatrix} a_0(x) & 0 & 0 & \dots & 0\\ a_1(x) & a_0(x) & 0 & \dots & 0\\ a_2(x) & a_1(x) & a_0(x) & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & 0\\ a_n(x) & a_{n-1}(x) & a_{n-2}(x) & \dots & a_0(x)\end{pmatrix},$$

acting on $e^{\xi(x,z)}$, respectively $e^{-\xi(x,z)}$. The following fundamental lemma will be useful.

Lemma 2.1. *Let $P(x,\partial)$ and $Q(x,\partial)$ be matrix pseudo-differential operators, then*

$$\mathrm{Res}_z\,P(x,z)e^{\xi(x,z)}Q(y,z)e^{-\xi(y,z)}=\mathbf{0}$$

if and only if

$$\left(P(x,\partial)Q(x,\partial)^*\right)_-=\mathbf{0}.$$

Here $P_-=P-P_+$, which is the pseudo-differential operator minus its differential part P_+ and * stands for the adjoint of a pseudo-differential operator, which is $(A(x)\partial^m)^*=(-\partial)^m A(x)$.

For a proof of this lemma see Corollary 4.1 of [5]. Note that we do not have to transpose the matrix $A(x)$ in the adjoint like in [5] because the coefficients of the pseudo-differential operators commute. Using the above fundamental Lemma 2.1, one deduces from (2.9) that

$$\left(P^+(m,x,\partial)P^-(m,x,\partial)^*\right)_-=0. \tag{2.11}$$

Since both

$$P^{\pm}(m, x, \partial) = I + \text{lower order terms in } \partial,$$

one gets that

$$P^{+}(m, x, \partial)P^{-}(m, x, \partial)^{*} = I$$

and thus

$$P^{-}(m, x, \partial)^{*} = P^{+}(m, x, \partial)^{-1}. \tag{2.12}$$

Now differentiating the first component of the tensor product of (2.9) with respect to x_j using again Lemma 2.1 and (2.12) one obtains the Sato–Wilson equations

$$\frac{\partial P^{+}(m, x, \partial)}{\partial x_j} = -\left(P^{+}(m, x, \partial)\partial^{j} P^{+}(m, x, \partial)^{-1}\right)_{-} P^{+}(m, x, \partial), \tag{2.13}$$

where we write ∂ for $I\partial$. Now introduce the Lax operator

$$L(m, x, \partial) = P^{+}(m, x, \partial)\partial P^{+}(m, x, \partial)^{-1}, \tag{2.14}$$

then (2.13) is equivalent to

$$L(m, x, \partial)\,\Psi^{+}(m, x, z) = z\Psi^{+}(m, x, z),$$
$$\frac{\partial \Psi^{+}(m, x, \partial)}{\partial x_j} = \left(L(m, x, \partial)^{j}\right)_{+} \Psi^{+}(m, x, z).$$

It is also straightforward to deduce from (2.13) the Lax equations

$$\frac{\partial L(m, x, \partial)}{\partial x_j} = \left[\left(L(m, x, \partial)^{j}\right)_{+}, L(m, x, \partial)\right] \tag{2.15}$$

and the Zakharov–Shabat equations

$$\frac{\partial \left(L(m, x, \partial)^{k}\right)_{+}}{\partial x_j} - \frac{\partial \left(L(m, x, \partial)^{j}\right)_{+}}{\partial x_k} = \left[\left(L(m, x, \partial)^{j}\right)_{+}, \left(L(m, x, \partial)^{k}\right)_{+}\right]. \tag{2.16}$$

3. Reductions

Fix a positive integer k and consider the case where

$$\frac{\partial \tau^{(m)}(x)}{\partial x_k} = \mu\tau^{(m)}(x), \qquad \mu \in \mathbb{C}.$$

Then also

$$\frac{\partial T^{(m)}(x)}{\partial x_k} = \mu T^{(m)}(x)$$

and

$$\frac{\partial \Psi^{\pm}(m, x, z)}{\partial x_k} = z^{\pm k}\Psi^{\pm}(m, x, z),$$

or equivalently

$$\frac{\partial P^{\pm}(m, x, \partial)}{\partial x_k} = 0.$$

Using the Sato–Wilson equations (2.13) we find that

$$L(m,x,\partial)^k = \left(L(m,x,\partial)^k\right)_+$$

and thus also

$$\frac{\partial P^{\pm}(m,x,\partial)}{\partial x_j} = 0 \quad \text{and} \quad \frac{\partial L(m,x,\partial)}{\partial x_j} = 0, \quad j = k, 2k, 3k, \ldots.$$

Now consider the case $n = 1$ and $k = 2$, then this case must be related to the coupled KdV equation. In this case

$$T(x) = T^{(0)}(x) = \begin{pmatrix} \tau_0(x) & 0 \\ \tau_1(x) & \tau_0(x) \end{pmatrix} \quad \text{and} \quad (T(x))^{-1} = \begin{pmatrix} \frac{1}{\tau_0(x)} & 0 \\ -\frac{\tau_1(x)}{\tau_0^2(x)} & \frac{1}{\tau_0(x)} \end{pmatrix}$$

and

$$\begin{aligned} P(x,\partial) &= P^+(0,x,\partial) = I + W_1(x)\partial^{-1} + W_2(x)\partial^{-2} + W_3(x)\partial^{-3} + \cdots \\ &= (T(x))^{-1}\left(T(x) - \frac{\partial T(x)}{\partial x_1}\partial^{-1} + \frac{1}{2}\left(\frac{\partial^2 T(x)}{\partial x_1^2} - \mu T(x)\right)\partial^{-2}\right. \\ &\quad \left. - \frac{1}{6}\left(\frac{\partial^3 T(x)}{\partial x_1^3} - 3\mu\frac{\partial T(x)}{\partial x_1} + 2\frac{\partial T(x)}{\partial x_3}\right)\partial^{-3} + \cdots\right). \end{aligned}$$

Then

$$W_1(x) = -(T(x))^{-1}\frac{\partial T(x)}{\partial x_1}, \qquad W_2(x) = \frac{1}{2}\left((W_1(x))^2 - \frac{\partial W_1(x)}{\partial x_1} - \mu\right).$$

Writing $W'(x) = \frac{\partial W(x)}{\partial x_1}$ and still ∂ for $I\partial$, one has:

$$\begin{aligned} L^2(x,\partial) &= L^2(0,x,\partial) = \partial^2 - 2W_1'(x), \\ \left(L^3(x,\partial)\right)_+ &= \partial^3 - 3W_1'(x)\partial + 3W_1(x)W_1'(x) - 3W_2'(x) - 3W_1''(x) \\ &= \partial^3 - 3W_1'(x)\partial + 3W_1(x)W_1'(x) - \frac{3}{2}\left((W_1(x))^2 - W_1'(x) - \mu\right)' - 3W_1''(x) \\ &= \partial^3 - 3W_1'(x)\partial - \frac{3}{2}W_1''(x). \end{aligned}$$

If we now write

$$\mathcal{L} = L^2(x,\partial) = \partial^2 + V(x) \quad \text{and} \quad \mathcal{B} = -(L^3(x,\partial))_+ = -\partial^3 - \frac{3}{2}V(x)\partial - \frac{3}{4}V'(x),$$

then the Lax equation (2.15) turns into

$$\frac{\partial \mathcal{L}}{\partial x_3} = [\mathcal{L}, \mathcal{B}],$$

which is exactly the Lax equation for the coupled KdV equation that appears in [1].

4. Bäcklund–Darboux transformations

It is our aim to obtain so-called elementary Bäcklund–Darboux transformations for this hierarchy. The key ingredients in the construction of these transformations are the commutation relations (1.9). We let

$$\sum_{i=0}^{n} g_i(w)\psi^{\pm(i)}(w) \otimes 1$$

act on (2.1) and this gives

$$\mathrm{Res}_z \sum_{i=0}^{n} g_i(w)\psi^{\pm(i)}(w) \sum_{k=0}^{n}\sum_{\ell=0}^{k} \frac{1}{k+1} \psi^{+(k-\ell)}(z)\tau^{(m)} \otimes_{\mathbb{C}^{(n)}(\lambda)} \psi^{-(\ell)}(z)\tau^{(m)} = 0. \tag{4.1}$$

Using the commutation relations (1.9) we obtain in the + case

$$\mathrm{Res}_z \sum_{i=0}^{n} g_i(w) \sum_{k=0}^{n}\sum_{\ell=0}^{k} \frac{1}{k+1} \psi^{+(k-\ell)}(z)\psi^{+(i)}(w)\tau^{(m)} \otimes_{\mathbb{C}^{(n)}(\lambda)} \psi^{-(\ell)}(z)\tau^{(m)} = 0 \tag{4.2}$$

and in the − case

$$\mathrm{Res}_z \sum_{i=0}^{n} g_i(w) \sum_{k=0}^{n}\sum_{\ell=0}^{k} \frac{1}{k+1} \left(\Lambda^{i+k-\ell}\delta(w-z) - \psi^{+(k-\ell)}(z)\psi^{-(i)}(w)\right) \tau^{(m)} \otimes_{\mathbb{C}^{(n)}(\lambda)} \psi^{-(\ell)}(z)\tau^{(m)} = 0. \tag{4.3}$$

Note that we could have replaced in both equations $\sum_{k=0}^{n}$ by $\sum_{k=0}^{i}$, we will however not do that. Next we let

$$1 \otimes \sum_{j=0}^{n} g_j(y)\psi^{+(j)}(y), \qquad 1 \otimes \sum_{j=0}^{n} g_j(y)\psi^{-(j)}(y)$$

act on (4.2) and (4.3) respectively. Again using (1.9), this gives

$$\mathrm{Res}_z \sum_{i,j=0}^{n} g_i(w)g_j(y) \sum_{k=0}^{n}\sum_{\ell=0}^{k} \frac{1}{k+1} \psi^{+(k-\ell)}(z)\psi^{+(i)}(w)\tau^{(m)} \otimes_{\mathbb{C}^{(n)}(\lambda)} \left(\Lambda^{j+\ell}\delta(y-z) - \psi^{-(\ell)}(z)\psi^{-(j)}(y)\right)\tau^{(m)} = 0 \tag{4.4}$$

and

$$\mathrm{Res}_z \sum_{i,j=0}^{n} g_i(w)g_j(y) \sum_{k=0}^{n}\sum_{\ell=0}^{k} \frac{1}{k+1} \left(\Lambda^{i+k-\ell}\delta(w-z) - \psi^{+(k-\ell)}(z)\psi^{-(i)}(w)\right)\tau^{(m)} \otimes_{\mathbb{C}^{(n)}(\lambda)} \psi^{-(\ell)}(z)\psi^{-(j)}(y)\tau^{(m)} = 0. \tag{4.5}$$

We now take the Res_w of the expressions (4.2), (4.3) and $\mathrm{Res}_w\mathrm{Res}_y$ of (4.4), (4.5) and define

$$\tau^{(m\pm1)} = \mathrm{Res}_w \sum_{i=0}^{n} g_i(w)\psi^{\pm(i)}(w)\tau^{(m)}.$$

Then (4.2) turns into the first equation of (4.6), which is

$$\begin{aligned}
&\mathrm{Res}_z \sum_{k=0}^{n}\sum_{\ell=0}^{k} \frac{1}{k+1} \psi^{+(k-\ell)}(z)\tau^{(m+1)} \otimes_{\mathbb{C}^{(n)}(\lambda)} \psi^{-(\ell)}(z)\tau^{(m)} = 0,\\
&\mathrm{Res}_z \sum_{k=0}^{n}\sum_{\ell=0}^{k} \frac{1}{k+1} \psi^{+(k-\ell)}(z)\tau^{(m-1)} \otimes_{\mathbb{C}^{(n)}(\lambda)} \psi^{-(\ell)}(z)\tau^{(m)} = \sum_{k=0}^{n} \lambda^k \tau^{(m)} \otimes_{\mathbb{C}^{(n)}(\lambda)} \tau^{(m-1)},\\
&\mathrm{Res}_z \sum_{k=0}^{n}\sum_{\ell=0}^{k} \frac{1}{k+1} \psi^{+(k-\ell)}(z)\tau^{(m\pm1)} \otimes_{\mathbb{C}^{(n)}(\lambda)} \psi^{-(\ell)}(z)\tau^{(m\pm1)} = 0.
\end{aligned} \tag{4.6}$$

Using the fact that

$$\mathrm{Res}_w \sum_{i=0}^{n} g_i(w) \sum_{\ell=0}^{k} \Lambda^{i+k-\ell}\delta(w-z)\tau^{(m)} \otimes_{\mathbb{C}^{(n)}(\lambda)} \psi^{-(\ell)}(z)\tau^{(m)} = \lambda^k \tau^{(m)} \otimes_{\mathbb{C}^{(n)}(\lambda)} \tau^{(m-1)}$$

and taking Res_w of (4.3), we obtain the second equation of (4.6). Since

$$\mathrm{Res}_w \mathrm{Res}_y \sum_{i,j=0}^{n} g_i(w) g_j(y) \psi^{-(i)}(w) \psi^{-(j)}(y) = 0,$$

one has

$$\mathrm{Res}_w \mathrm{Res}_y \mathrm{Res}_z \sum_{i,j=0}^{n} g_i(w) g_j(y) \sum_{k=0}^{n} \sum_{\ell=0}^{k} \frac{\Lambda^{i+k-\ell}}{k+1} \delta(w-z) \tau^{(m)} \otimes_{\mathbb{C}^{(n)}(\lambda)} \psi^{-(\ell)}(z) \psi^{-(j)}(y) \tau^{(m)}$$
$$= \sum_{k=0}^{n} \lambda^k \tau^{(m)} \otimes_{\mathbb{C}^{(n)}(\lambda)} \mathrm{Res}_w \mathrm{Res}_y \sum_{i,j=0}^{n} g_i(w) g_j(y) \psi^{-(i)}(w) \psi^{-(j)}(y) \tau^{(m)} = 0.$$

Using this and taking $\mathrm{Res}_w \mathrm{Res}_y$ of (4.5) one obtains the minus part of the last equation of (4.6). The plus part follows in a similar way from (4.4). We thus obtained in particular that both $\tau^{(m+1)}$ and $\tau^{(m-1)}$ satisfy Eq. (2.1). If we write again

$$\tau^{(m)} = \sum_{j=0}^{n} \tau_j^{(m)} \lambda^j q^m$$

we find that

$$\tau_j^{(m\pm1)} = \sum_{i=0}^{j} \mathrm{Res}_w \, g_i(w) w^{\pm m}$$
$$= \mathrm{Res}_w \sum_{i=p}^{j+p} g_{j+p-i}(w) w^{\pm m} \mathrm{e}^{\pm\xi(x,w)} \mathrm{e}^{\mp\eta(x,w)} \tau_{i-p}^{(m)}$$

and thus also (see (2.4) and (2.6)):

$$\begin{aligned} T^{(m\pm1)}(x)_{p+j,p} = \tau_j^{(m\pm1)} &= \mathrm{Res}_w \sum_{i=p}^{j+p} g_{j+p-i}(w) \Phi^{\pm}(m,x,w)_{ip} \\ &= \mathrm{Res}_w \sum_{i=0}^{j+p} g_{j+p-i}(w) \Phi^{\pm}(m,x,w)_{ip}. \end{aligned} \tag{4.7}$$

Hence,

$$T^{(m\pm1)}(x) = \mathrm{Res}_w \, G(w) \Phi^{\pm}(m,x,w),$$

with $G(w)$ defined by

$$G(w) = \begin{pmatrix} g_0(w) & 0 & 0 & \ldots & 0 \\ g_1(w) & g_0(w) & 0 & \ldots & 0 \\ g_2(w) & g_1(w) & g_0(w) & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ g_n(w) & g_{n-1}(w) & g_{n-2}(w) & \ldots & g_0(w) \end{pmatrix}. \tag{4.8}$$

Note that also

$$T^{(m\pm1)}(x) = \mathrm{Res}_w \, \Phi^{\pm}(m,x,w) G(w),$$

and thus

$$T^{(m\pm1)}(x) = \mathrm{Res}_w \, T^{(m)}(x) \Psi^{\pm}(m,x,w) G(w)$$

$$
\begin{aligned}
&= \mathrm{Res}_w\, G(w)\, \Psi^{\pm}(m,x,w) T^{(m)}(x) \\
&= Q^{\pm}(m,x) T^{(m)}(x),
\end{aligned}
$$

which suggests the main theorem of this paper:

Theorem 4.1. *Let $T^{(m)}(x)$ be a tau matrix, i.e., satisfy (2.5), $\Psi^{\pm}(m,x,z)$ be the corresponding (adjoint) wave matrix satisfying (2.9) and $G(w)$ be given by (4.8) such that*

$$
Q^{\pm}(m,x) = \mathrm{Res}_w\, G(w)\, \Psi^{\pm}(m,x,w) \tag{4.9}
$$

is well defined. Then

$$
T^{(m\pm 1)}(x) = Q^{\pm}(m,x) T^{(m)}(x) \tag{4.10}
$$

satisfies again (2.5) and the corresponding (adjoint) wave matrices are given by

$$
\begin{aligned}
\Psi^{+}(m\pm 1,x,z) &= \left(Q^{\pm}(m,x)\right)^{\pm 1} \partial^{\pm 1} \left(Q^{\pm}(m,x)\right)^{\mp 1} \Psi^{+}(m,x,z), \\
\Psi^{-}(m\pm 1,x,z) &= -\left(Q^{\pm}(m,x)\right)^{\mp 1} \partial^{\mp 1} \left(Q^{\pm}(m,x)\right)^{\pm 1} \Psi^{-}(m,x,z).
\end{aligned} \tag{4.11}
$$

We call $Q^{+}(m,x)$ an eigenfunction and $Q^{-}(m,x)$ an adjoint eigenfunction.

We proceed with the rest of the proof of the theorem. The first equation of (4.6) gives the equation

$$
\mathrm{Res}_z\, \Psi^{+}(m+1,x,z)\, \Psi^{-}(m,x',z) = \mathbf{0}.
$$

Using Lemma 2.1 and (2.12) one gets that

$$
\left(P^{+}(m+1,x,\partial)\partial P^{+}(m,x,\partial)^{-1}\right)_{-} = \mathbf{0}
$$

or equivalently that

$$
\begin{aligned}
P^{+}(m+1,x,\partial)\partial P^{+}(m,x,\partial)^{-1} &= \partial I - \left(T^{(m+1)}(x)\right)^{-1} \frac{\partial T^{(m+1)}(x)}{\partial x_1} + \left(T^{(m)}(x)\right)^{-1} \frac{\partial T^{(m)}(x)}{\partial x_1} \\
&= T^{(m)}(x) \left(T^{(m+1)}(x)\right)^{-1} \partial T^{(m+1)}(x) \left(T^{(m)}(x)\right)^{-1} \\
&= Q^{+}(m,x)\partial \left(Q^{+}(m,x)\right)^{-1},
\end{aligned} \tag{4.12}
$$

proving the formulas for $\Psi^{+}(m+1,x,z)$ of (4.11). The second equation of (4.6) is equivalent to

$$
\mathrm{Res}_z\, \Psi^{+}(m-1,x,z)\, \Psi^{-}(m,x',z) = T^{(m)}(x) T^{(m-1)}(x').
$$

Using Lemma 2.1 again and (2.12) one gets that

$$
\begin{aligned}
\left(P^{+}(m-1,x,\partial)\partial^{-1} P^{+}(m,x,\partial)^{-1}\right)_{-} &= \left(T^{(m-1)}(x)\right)^{-1} T^{(m)}(x)\partial^{-1} \left(T^{(m)}(x)\right)^{-1} T^{(m-1)}(x) \\
&= \left(Q^{-}(m,x)\right)^{-1} \partial^{-1} Q^{-}(m,x),
\end{aligned}
$$

or equivalently that

$$
P^{+}(m-1,x,\partial)\partial^{-1} P^{+}(m,x,\partial)^{-1} = \left(Q^{-}(m,x)\right)^{-1} \partial^{-1} Q^{-}(m,x),
$$

which gives the formula for $\Psi^{+}(m-1,x,z)$ in (4.11). From the first equation of (4.11), using (2.12) one can easily deduce the last one.

5. The scalar formulation

To generate solutions from the vacuum one wants to let several of these Bäcklund–Darboux transformations act. The matrix formulation in that case is not so convenient. We therefore also present another formulation. We return to

$$\tau^{(m)}(x) = \sum_{j=1}^{n} \tau_j^{(m)}(x)\lambda^j$$

over the ring $\mathbb{C}^{(n)}(\lambda)$. Clearly if $\tau_0^{(m)}(x) \neq 0$ the element $\tau^{(m)}(x)$ has an inverse, viz.,

$$\left(\tau^{(m)}(x)\right)^{-1} = \frac{1}{\tau_0^{(m)}(x)} \left(1 - \frac{\tau_1^{(m)}(x)}{\tau_0^{(m)}(x)}\lambda + \left(\left(\frac{\tau_1^{(m)}(x)}{\tau_0^{(m)}(x)}\right)^2 - \frac{\tau_2^{(m)}(x)}{\tau_0^{(m)}(x)}\right)\lambda^2 + \cdots\right),$$

we define the (adjoint) wave functions to be

$$\tilde{\psi}^{\pm(m)}(x,z) = \left(\tau^{(m)}(x)\right)^{-1} z^{\pm} \mathrm{e}^{\pm\xi(x,z)} \mathrm{e}^{\mp\eta(x,z)} \tau^{(m)}(x), \tag{5.1}$$

which we write as pseudo-differential operators

$$P^{\pm(m)}(x,\partial) = \sum_{i=0}^{n} P_i^{\pm(m)}(x,\partial)\lambda^i$$

acting on $\mathrm{e}^{\pm\xi(x,z)}$, i.e.,

$$\tilde{\psi}^{\pm(m)}(x,z) = \sum_{i=0}^{n} \tilde{\psi}_i^{\pm(m)}(x,z)\lambda^i = P^{\pm(m)}(x,\partial)\partial^{\pm m} \mathrm{e}^{\pm\xi(x,z)}.$$

Let

$$g(z) = \sum_{i=1}^{n} g_i(z)\lambda^i$$

be such that

$$q^{\pm(m)}(x) = \sum_{i=0}^{n} q_i^{\pm(m)}(x)\lambda^i = \mathrm{Res}_z\, g(z)\tilde{\psi}^{\pm(m)}(x,z)$$

is well defined, then from the calculations of the previous section it is clear that one has the following:

Theorem 5.1. *The element*

$$\tau^{(m\pm1)}(x) = q^{\pm(m)}(x)\tau^{(m)}(x)$$

is again a tau function and the corresponding wave functions are given by

$$\begin{aligned}
\tilde{\psi}^{+(m\pm1)}(x,z) &= \left(q^{\pm(m)}(x)\right)^{\pm1} \partial^{\pm1} \left(q^{\pm(m)}(x)\right)^{\mp1} \tilde{\psi}^{+(m)}(x,z),\\
\tilde{\psi}^{-(m\pm1)}(x,z) &= -\left(q^{\pm(m)}(x)\right)^{\mp1} \partial^{\mp1} \left(q^{\pm(m)}(x)\right)^{\pm1} \tilde{\psi}^{-(m)}(x,z).
\end{aligned} \tag{5.2}$$

As a consequence, see e.g. [2], one has the following

Corollary 5.1. *Let $g^k(z) = \sum_{i=1}^{n} g_i^k(z)\lambda^i$ for $k = 1, 2, \ldots, N$ and let*

$$q^{\pm(m)k}(x) = \mathrm{Res}_z\, g^k(z)\tilde{\psi}^{\pm(m)}(x,z)$$

then

$$\tau^{(m+N)}(x) = W\left(q^{+(m)1}(x), q^{+(m)2}(x), \dots, q^{+(m)N}(x)\right)\tau^{(m)}(x)$$

is again a tau function and

$$\tilde\psi^{+(m+N)}(x,z) = \left(W\left(q^{+(m)1}(x), \dots, q^{+(m)N}(x)\right)\right)^{-1} W\left(q^{+(m)1}(x), \dots, q^{+(m)N,}(x), \tilde\psi^{+(m)}(x,z)\right)$$

is the corresponding wave function. Here

$$W(f_1, f_2, \dots, f_N) = \det\left(\frac{\partial^{i-1} f_j}{\partial x_1^{i-1}}\right)_{1\le i,j\le N}$$

is the Wronskian determinant.

In particular we can start with

$$\tau^{(0)}(x) = 1$$

and thus

$$\tilde\psi^{+(0)}(x,z) = e^{\xi(x,z)}.$$

Now choose non-negative integers μ_{ij} with $i = 1, 2, \dots, N$ and $j = 0, 1, \dots, n$ and define

$$g^k(z) = \sum_{i=0}^{n} z^{-\mu_{ki}-1},$$

then

$$q^{+(0)k}(x) = \sum_{i=0}^{n} S_{\mu_{ki}}(x)\lambda^k,$$

where $S_j(x)$ are the elementary Schur functions defined by

$$\sum_{j\in\mathbb{Z}} S_j(x) z^j = e^{\xi(x,z)}.$$

Then

$$\tau_j^{(N)}(x) = \sum_{i_r\ge 0,\ i_1+i_2+\cdots+i_N=j} W\left(S_{\mu_{1i_1}}(x), S_{\mu_{2i_2}}(x), \dots, S_{\mu_{Ni_N}}(x)\right).$$

For certain choices of integers $\tau_0^{(N)}(x) = S_{\lambda_1,\lambda_2,\dots,\lambda_N}(x)$ the Schur function is related to the partition

$$(\lambda_1, \lambda_2, \dots, \lambda_N), \quad \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_N,$$

viz. if we choose

$$\mu_{10} = \lambda_1 + N - 1, \mu_{20} = \lambda_2 + N - 2, \dots \ge \mu_{N0} = \lambda_N.$$

In particular if one chooses $\mu_{ij} = \lambda_{ij} + N - i$ with all $\lambda_{ij} > 0$ and

$$\lambda_{ij} \ge \lambda_{i+1,k} \quad \text{for all } 0 \le j,k \le n$$

then

$$\tau_j^{(N)}(x) = \sum_{i_r\ge 0,\ i_1+i_2+\cdots+i_N=j} S_{\lambda_{1i_1},\lambda_{2i_2},\dots,\lambda_{Ni_N}}(x).$$

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References

[1] P. Casati, G. Ortenzi, New integrable hierarchies from vertex operator representations of polynomial Lie algebras, J. Geom. Phys. 56 (3) (2006) 418–449.
[2] G.F. Helminck, J.W. van de Leur, Geometric Bäcklund–Darboux transformations for the KP hierarchy, Publ. Res. Inst. Math. Sci. 37 (4) (2001) 479–519.
[3] R. Hirota, X. Hu, X. Tang, A vector potential KdV equation and vector Ito equation: Soliton solutions, bilinear Bäcklund transformations and Lax pairs, J. Math. Anal. Appl. 288 (1) (2003) 326–348.
[4] V.G. Kac, Infinite Dimensional Lie Algebras, 3rd edition, Cambridge University Press, Cambridge, 1990.
[5] V.G. Kac, J.W. van de Leur, The n-component KP hierarchy and representation theory, J. Math. Phys. 44 (8) (2003) 3245–3293.
[6] S. Kakei, Dressing method and the coupled KP hierarchy, Phys. Lett. A 264 (6) (2000) 449–458.
[7] S.Yu. Sakovich, A note in the Painlevé property of coupled KdV equation, arXiv:nlin.SI/0402004.